Math 255A' Lecture 4 Notes

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1 Finite Dimensional Normed Spaces, Quotients, Products, and Dual Spaces

1.1 Norms on finite dimensional space

Theorem 1.1. *Let X be a normed space. If $\dim X < \infty$, then any two norms on X are equivalent.*

Proof. We can assume $X = \mathbb{F}^n$. If $\|\cdot\|$ is a mystery norm, we show that $\|\cdot\|$ is equivalent to the ℓ^1 norm $|x| = \sum_{i=1}^n |x_i|$.

Step 1: Let $e_1, \dots, e_n$ be the standard orthonormal basis of $\mathbb{F}^n$. Then let $M := \max_{1 \leq i \leq n} \|e_i\|$. Then

$$\|x\| = \left\| \sum_{i=1}^n x_i e_i \right\| \leq \sum_i |x_i| \|e_i\| \leq M|x|.$$

Step 1.5: This shows that $\mathrm{Id} : \mathbb{F}^n \to \mathbb{F}^n$ is continuous from $|\cdot|$ to $\|\cdot\|$. So $\{x : |x| = 1\}$ is compact according to $\|\cdot\|$.

Step 2: So we get $\varepsilon > 0$ such that any x with $|x| = 1$ has $\|x\| \geq \varepsilon$. So $\{\|\cdot\|/\varepsilon < 1\} \subseteq \{|\cdot| < 1\}$. That is, $|\cdot| \leq (1/\varepsilon)\|\cdot\|$. $\square$

Remark 1.1. A result called John's theorem gives explicit constants dependent on n.[1]

Corollary 1.1. *Any finite dimensional subspace of a normed space is closed.*

Corollary 1.2. *Let X, Y be a normed spaces with $\dim X < \infty$. Then if $T : X \to Y$ is linear, it must be continuous.*

Proof. $\|x\|_X + \|Tx\|_Y$ is a norm for X, so there is a constant $M < \infty$ such that $\|x\|_X + \|Tx\|_Y \leq M\|x\|_X$. $\square$

[1] Check out the proof of this one!

1.2 Quotients in normed spaces

Let X be a normed space over $\mathbb{F}$ with a subspace M. Linear algebra tells you that the quotient $X/M = \{x + M : x \in X\}$ is a vector space.

Definition 1.1. The quotient space X/M has the **quotient seminorm** $\|x + M\| := \inf\{\|x - y\| : y \in M\} = \operatorname{dist}(x, M)$.

Lemma 1.1. *The quotient seminorm is a norm if and only if M is closed.*

Definition 1.2. The **quotient map** is the map $Q : X \to X/M$ given by $x \mapsto x + M$.

Theorem 1.2. *The quotient has the following properties:*

1. $\|Qx\| \leq \|x\|$ *for all* $x \in X$.
2. *If X is a Banach space and M is closed, then X/M is a Banach space.*
3. $U \subseteq X/M$ *is open if and only if* $Q^{-1}(U)$ *is open in* X.
4. *Q is an open mapping.*

Proof. 1. Since $0 \in M$, $\|x + M\| \leq \|x + 0\| = \|x\|$.

2. Suppose $(x_n + M)_n$ is Cauchy in X/M. Then there is a subsequence $(x_{n_i} + M)_i$ such that $\|x_{n_i} - x_{n_{i+1}} + M\| < 2^{-i}$ for all i. Then there is a $y_i \in M + (x_{n_i} - x_{n_{i+1}})$ such that $\|y_i\| < 2^{-i}$. Now $x_{n_2} \in x_{n_1} + y_1 + M$, $x_{n_3} \in x_{n_1} + y_1 + y_2 + M$, and so on, giving us $x_{n_{i+1}} \in x_{n_1} + y_1 + \cdots y_i + M$, where $x_{n_i} + y_1 + \cdots y_i$ is a Cauchy sequence in X.

   Now suppose that $x_{n_i} + y_1 + \cdots y_i \to z$. Then $\|x_{n_{i+1}} - z + M\| \to 0$. Then $x_n + M \to z + M$ in X/M

3. This is a rephrasing of (4).

4. Continuity follows from part(1). If $U \subseteq X$ is open, $x \in U$, and $B(x, \varepsilon) \subseteq U$, then any $y + M \in B_{X/M}(x + M, \varepsilon) = Q(B(x, \varepsilon)) \subseteq Q(U)$.

□

Definition 1.3. We use $M \leq X$ to say that M is a *closed* subspace of X.

Theorem 1.3. *If X is a normed space, $M \leq X$, and N is any finite dimensional subspace, then $M + N = \{x + y : x \in M, y \in N\}$ is closed.*

Proof. Observe that $M + N = Q^{-1}(Q(N))$. $Q(N)$ is finite dimensional, so it is closed. Q is continuous, so $Q^{-1}(Q(N))$ is closed. □

Remark 1.2. This is surprisingly tricky to prove without using the quotient X/M.

1.3 Products of normed spaces

If we have a general family $(X_i)_{i\in I}$ of normed spaces, there is no canonical norm on the product. We may define notions of product by considering various subspaces of $\prod_{i\in I} X_i$.

Example 1.1. Fix $1 \leq p < \infty$. The ℓ^p**-direct sum** $\bigoplus_p X_i = \{(x_i)_{i\in I} \in \prod_i X_i : \sqrt{\sum_i \|x_i\|_i^p} < \infty\}$ is a normed space with the norm $\|(x_i)_i := \sqrt{\sum_i \|x_i\|_i^p}$.

Example 1.2. The ℓ^∞**-direct sum** $\bigoplus_\infty X_i = \{(x_i)_{i\in I} \in \prod_i X_i : \sup_i \|x_i\|_i < \infty\}$ is a normed space with the norm $\|(x_i)_i := \sup_i \|x_i\|_i < \infty$.

Example 1.3. If $I = \mathbb{N}$, we also have $\bigoplus_0 X_i = \{(x_i)_{i\in I} \in \prod_i X_i : \|x_i\|_i \to 0\}$.

Proposition 1.1. *1. For each of these notions of product X, X is complete if and only if X_i is complete for all i.*

2. $X \to X/\{(x_i)_{i\in I} : x_i = 0\}$ is an isometry to X_i.

3. Each coordinate projection $X \to X_i$ has norm 1 and is open.

1.4 Dual spaces

Definition 1.4. The **dual** of X is the space $X^* := \mathcal{B}(X, \mathbb{F})$ of bounded linear functionals. The **dual norm** is $\|L\|_* := \sup\{|L(x)| : \|x\| = 1\}$.

Proposition 1.2. *If Y is complete, $\mathcal{B}(X, Y)$ is complete.*

Corollary 1.3. *X^* is a Banach space.*

Here is a proof of this fact independent of the general fact about operators.

Proof. Let $L \in X^*$, and consider $L|_B$ restricted to the closed unit ball. Then $L|_B \in C_b(B)$. So the map ρ sending $L \mapsto L|_B$ gives us that $\rho(X^*)$ is a lienar subspace of $C_b(X)$. Moreover, $\rho(X^*)$ is closed. Since $C_b(B)$ is complete, so is □

Example 1.4. Let $X = c_0 = \{(x_i)_{i\in\mathbb{N}} \in \mathbb{R}^\mathbb{N} : x_i \to 0\}$. Then $L(x_1, x_2, \dots) = x_1$ is a linear functional.

Let e_i be the vector with all 0s but a 1 in the i-th coordinate. Then $\{e_1, e_2, \dots,\} \cup \{(1, 1/2, 1/3.1/4, \dots)\}$ is linearly independent. So there exists a linear functional $L : c_0 \to \mathbb{R}$ such that $L(e_i) = 0$ for all i and $L(1, 1/2, 1/3, \dots) = 1$. This L is not continuous.